with a stagnation point, i.e., one has to choose γ so that $G'(w^*)=0$, for $w^*=g^{-1}(z^*)$.

3) Transplant back to get the solution $F(z)$ of the original flow problem. From Eq. (1), there follows $F(z)=G[g^{-1}(z)]$.

4) Compute force L exerted on the profile according to Blasius' theorem,

$$L=-\frac{i\rho}{2}\int_\Gamma |q|^2 dz=-\frac{i\rho}{2}\int_\Gamma \overline{[F'(z)]^2 dz} \tag{8}$$

where Γ is any positively oriented curve in D encircling the profile and ρ the density (see Ref. 6, p. 366).

V. Numerical Results

In Ref. 1, two approximate formulas are given for the lift L of a flat-plate airfoil inclined at an arbitrary angle α to an incompressible stream between parallel walls. L will depend on the height h of the tunnel, the chord c, and the angle of incidence α of the airfoil and on the distance d of its midpoint from the floor of the tunnel.

For the midchord of the plate in the centerline of the tunnel, i.e., $d=0.5h$, Tomotika (see Ref. 1, p. 34) derives

$$\frac{L}{L_0}=1+\frac{\pi^2}{24}(1+\sin^2\alpha)\left(\frac{c}{h}\right)^2 -\frac{\pi^4}{7680}(11-53\sin^2\alpha-22\sin^4\alpha)\left(\frac{c}{h}\right)^4+\mathcal{O}\left[\left(\frac{c}{h}\right)^6\right] \tag{9}$$

where $L_0=i\rho|q_\infty|^2 c\pi\sin\alpha$ is the freestream lift.

We compare now the tunnel to free-air ratios $(L/L_0)_T$ estimated by Eq. (9) with our exact values L/L_0 computed by Eq. (8).

In Fig. 2, the values $\log_{10}[L/L_0-(L/L_0)_T]$ for $\alpha=0.05$ and 20 deg are plotted against c/h, which makes sense, as $(L/L_0)_T$ was always smaller than (L/L_0). When $\alpha=0.05$ deg and $c<0.33h$, the error of Eq. (9) is <0.0001, but it exceeds 0.01 when $c=0.75h$. For the corresponding values of $\alpha=20$ deg, the error is <0.0001 for $c<0.36h$, but it exceeds 0.001 when $c=0.53h$.

Havelock (Ref. 1, p. 34) gives a formula for L/L_0 for a plate whose midpoint is at an arbitrary distance d from the floor of the tunnel,

$$\frac{L}{L_0}=1-\frac{\pi}{2}\cot\left(\frac{\pi d}{h}\right)\sin\alpha\left(\frac{c}{h}\right)+\frac{\pi^2}{16}\left\{\left[\frac{2}{3}+\cot^2\left(\frac{\pi d}{h}\right)\right] +\left[\frac{2}{3}+3\cot^2\left(\frac{\pi d}{h}\right)\right]\sin^2\alpha\right\}\left(\frac{c}{h}\right)^2+\mathcal{O}\left[\left(\frac{c}{h}\right)^3\right] \tag{10}$$

Figure 3 shows the variation of $L/L_0-(L/L_0)_H$ with the ratio d/h where $(L/L_0)_H$ is computed by Eq. (10) for selected values of α and $c=0.2h$. Even for relatively small chord c (i.e., $c=0.2h$), Havelock's formula is not very accurate for $\alpha=20$ deg and $d=0.8h$: Eq. (10) yields $L/L_0=1.22941$, whereas we obtained $L/L_0=1.23857$.

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Approximate Analysis of Deflections and Frequencies of Short Beams

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Introduction

MANY papers have been published on the analysis of stresses, deflections, and frequencies of short beams. For instance, considering the warping of a section, Murty[1,2] and Levinson[3,4] introduced the equations of motion and Rehfield and Murthy,[5] assuming the stress function as $\Sigma_m\Sigma_n a_{mn}x^m y^n$, determined the stress distribution of a beam subjected to a uniformly distributed load. Several questionable points in their methods and results were discussed in detail in Ref. 6.

To remove these defects, assuming the axial normal stress σ_x as $\Sigma_n y^n u_n(x)$, the author[6] obtained the shearing stress τ and normal stress σ_y in the transverse direction by the equilibrium conditions and determined u_n, using the minimum complementary energy principle. However, dynamical cases cannot be analyzed by this method.

The following method will be applied in this Note: The axial displacement u is assumed to be in the form of yu_1+Yu_2, where Y is a specified odd function with respect to y. In the first step, u_1 and u_2 are determined by solving the fundamental equations obtained from the equilibrium conditions. In the second step, on the assumption of $Y=\Sigma_m b_m y^m$, the coefficients b_m are determined as to minimize the total energy V for static loads and to satisfy the condition $V_s=V_p$ for dynamic loads, where V_s and V_p are the strain and potential energies, respectively. The latter condition is not satisfied in Levinson's method.[3]

Determination of Deflections and Stresses Under Statical Loads

In order to simplify the calculations, the case will be considered where a beam with a rectangular cross section is sub-

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jected to a uniformly distributed load q_0 and both its ends are clamped. Of course, general cases can also be analyzed by the following method. The origin is located at the midpoint of the beam.

The normal and shearing stresses σ_x and τ are

$$\sigma_x \Big/ \frac{E}{\ell} = \frac{\partial u}{\partial \xi}$$

$$\tau \Big/ \frac{G}{h} = \frac{\partial u}{\partial \eta} + r\,\frac{\partial v}{\partial \xi} \tag{1}$$

where u, v, 2ℓ, $2h$, E, and G are displacements in the axial and transverse directions, length, thickness, Young's modulus, and shear modulus, respectively, and where $\xi = x/\ell$, $\eta = y/h$, and $r = h/\ell$.

In this Note, the normal stress σ_y in the transverse direction and Poisson's ratio ν are neglected. The displacements u and v are assumed as

$$u = \eta u_1 + Y u_2 \qquad v = v \tag{2}$$

where u_1, u_2, and v are the functions with respect to ξ only, and Y is any specified odd function with respect to η. From the conditions $\tau = 0$ at $\eta = \pm 1$

$$u_1 = -\dot{Y}_1 u_2 - r v' \tag{3}$$

where ( )$'$ and ($\dot{}$) denote the differentiations with respect to ξ and η, where $\dot{Y}_1 = \dot{Y}(\eta = 1)$.

The shearing force Q and bending moment M are expressed as

$$Q = \int_{-h}^{h} \tau \, dy = G a_1 u_2$$

$$M = \int_{-h}^{h} \sigma_x y \, dy = E r h (a_2 u_2' - 2 r v''/3) \tag{4}$$

where $a_1 = \int_{-1}^{1} (\dot{Y} - \dot{Y}_1)\, d\eta$ and $a_2 = \int_{-1}^{1} (\eta Y - \eta^2 \dot{Y}_1)\, d\eta$.

The equilibrium equations are

$$Q' = -\ell q_0 \qquad Q\ell = M' \tag{5}$$

Substituting Eq. (4) into Eq. (5) gives

$$v'''' = q \tag{6}$$

where $q = 3 q_0 \ell / 2 E r^3$. Then, u_1 becomes

$$u_1 = (2 k a_3 r / 3 a_1) v''' - r v' \tag{7}$$

where $k = E r^2 / G$ and $a_3 = \dot{Y}_1$. As both ends of the beam are clamped, boundary conditions are

$$u_1 = v = 0 \qquad \text{at } \xi = \pm 1 \tag{8}$$

With the aid of Eq. (8), the displacement and stresses become

$$v/q = (1-\xi^2)^2/24 - (1-\xi^2) a_3 k / 3 a_1$$

$$u_1/q = (\xi - \xi^3) r/6, \qquad u_2/q = -2 r k \xi / 3 a_1 \tag{9}$$

$$\sigma_x / (E/\ell) = Y u_2' + \eta u_1'$$

$$\tau / (G/h) = (\dot{Y} - \dot{Y}_1) u_2 \tag{10}$$

We have only to assume Y as the odd function with respect to η. In his paper, Levinson[3] puts $Y = \eta^3$.

In the second step, the accuracy of solution will be improved. The total energy V is

$$V = \int_{-1}^{1} \int_{-1}^{1} \left(\frac{\sigma_x^2}{2E} + \frac{\tau^2}{2G} \right) \ell h \, d\eta \, d\xi - \int_{-1}^{1} q_0 v \ell \, d\xi \tag{11}$$

Substitution of Eqs. (9) and (10) into Eq. (11) and integration with respect to ξ gives

$$V \Big/ \frac{4Gr}{27} k^2 q^2 = \int_{-1}^{1} (\dot{Y}^2 - 2 a_3 \dot{Y} + 3 k Y^2)/a_1^2 \, d\eta$$

$$+ 2 a_3^2 / a_1^2 + 2 a_3 / a_1 - 1/10k \tag{12}$$

Now, Y will be assumed to be

$$Y = \sum_m b^m \eta^m \tag{13}$$

where m is the odd integer. Coefficients b_m can be determined by the conditions $\partial V / \partial b_m = 0$.

Determination of Frequencies

The frequencies of the beam can be determined in a similar way. In this case, the equations of motion are

$$Q' = -2 \rho h \ell \omega^2 v$$

$$Q\ell - M' = \rho h^2 \ell \omega^2 \int_{-1}^{1} u \eta \, d\eta \tag{14}$$

where ρ and ω are density and frequency of the beam, respectively.

Table 1 Relationships between v/v_{BE}, σ_x/σ_{xBE} V, and r for different functions Y

Y	r	1/4	1/6	1/10	1/15
η^5	v/v_{BE}	1.8125	1.3610	1.1300	1.0578
	σ_x/σ_{xBE}	1.0813	1.0361	1.0130	1.0058
	$V\Big/\frac{4Gr}{27}k^2q^2$	−1.3084	−2.0784	−4.5404	−9.3481
η^3	v/v_{BE}	1.9750	1.4332	1.1560	1.0693
	σ_x/σ_{xBE}	1.1625	1.0722	1.0260	1.0116
	$V\Big/\frac{4Gr}{27}k^2q^2$	−1.5067	−2.2807	−4.7448	−9.5532
$\eta^3 + b\eta^5$	v/v_{BE}	2.9002	1.9121	1.3438	1.1558
	σ_x/σ_{xBE}	1.6251	1.3117	1.1199	1.0547
	$V\Big/\frac{4Gr}{27}k^2q^2$	−2.0275	−2.9065	−5.4362	−10.2282
	b	−0.4253	−0.4345	−0.4392	−0.4407

Table 2 Relationship between $\lambda^{1/4}$ and r for different functions Y

Y \ r	1/4	1/6	1/10	1/15
η^5	4.0464	4.3576	4.5769	4.6588
η^3	3.9730	4.3102	4.5553	4.6483
$\eta^3 + b\eta^5$	3.7001	4.1035	4.4524	4.5962
b	−0.4108	−0.4167	−0.4173	−0.4180
	$\lambda_{BE}^{1/4} = 4.7330$			

Substitution of Eq. (4) into Eq. (14) gives

$$v = -\frac{24 G a_1}{\lambda E r^3} u_2'$$

$$Ga_1 u_2 = Er^2\left(a_2 u_2'' - \frac{2r}{3} v'''\right) = \frac{\lambda E r^4}{48}\left(a_2 u_2 - \frac{2r}{3} v'\right) \quad (15)$$

From Eq. (15)

$$u_2'''' + \left(\frac{Ea_2}{Ga_1} + \frac{1}{3}\right)\frac{\lambda r^2}{16} u_2'' + \frac{\lambda}{16}\left(\frac{\lambda E r^4 a_2}{48 Ga_1} - 1\right) u_2 = 0 \quad (16)$$

where $\lambda = 48\rho\omega^2\ell^4/Er^2$.

To simplify the calculations, frequencies for the modes symmetrical with respect to the midpoint of the beam will be considered. For the case $\lambda < 48Ga_1/Er^4 a_2$, u_2 becomes

$$u_2 = C_1 \sinh\alpha\xi + C_2 \sin\beta\xi \quad (17)$$

where

$$\frac{\alpha^2}{\beta^2} = \frac{\lambda r^2}{4}\left[\left\{\left(\frac{Ea_2}{8Ga_1} - \frac{1}{24}\right)^2 + \frac{1}{\lambda r^4}\right\}^{1/2} \mp \left(\frac{Ea_2}{8Ga_1} + \frac{1}{24}\right)\right]$$

and C_i are constants.

With the aid of Eq. (8), λ and u_2 can be determined from the following formulas:

$$(-a_3 + c\alpha^2)\beta \sinh\alpha \cos\beta + (a_3 + c\beta^2)\alpha \cosh\alpha \sin\beta = 0 \quad (18)$$

$$u_2 = C_2\left(-\frac{\beta\cos\beta}{\alpha\cosh\alpha}\sinh\alpha\xi + \sin\beta\xi\right) \quad (19)$$

where $c = 24Ga_1/\lambda Er^2$. The strain energy V_s and potential energy V_p are

$$V_s = \int_{-h}^{h}\int_{-\ell}^{\ell}\left(\frac{\sigma_x^2}{2E} + \frac{\tau^2}{2G}\right) dx\, dy$$

$$V_p = \frac{\rho\omega^2}{2}\int_{-h}^{h}\int_{-\ell}^{\ell}(u^2 + v^2)\, dx\, dy \quad (20)$$

Therefore,

$$V_s \bigg/ \frac{G}{2r} = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{Er^2}{G}\{(Y - \eta\dot{Y}_1)u_2' + c\eta u_2'''\}^2 + (\dot{Y} - \dot{Y}_1)^2 u_2^2\right) d\eta\, d\xi$$

$$V_p \frac{G}{2r} = \frac{\lambda E r^4}{48G}\left(\frac{2c^2}{r^2}\int_{-1}^{1} u'^2 d\xi + \int_{-1}^{1}\int_{-1}^{1}\{(Y - \eta\dot{Y}_1)u_2 + c\eta u_2''\}^2\right) d\eta\, d\xi \quad (21)$$

Coefficient b can be determined by the condition $V_s = V_p$.

Numerical Example and Discussion

As an example, the cases are treated where $Y = \eta^3$, η^5, and $\eta^3 + b\eta^5$. For the case where q_0 is applied statically, the relationships between r, deflection and outer-fiber stress at midpoint are listed in Table 1. The suffix BE indicates the value given by Bernoulli-Euler theory. The coefficient b becomes

$$b = (-54/35 + 11k/21)(220/63 - 13k33)$$

It is found that the values of deflection and outer-fiber stress become much larger than v_{BE} and σ_{xBE} as the value of r increases. For comparison, the value of V is listed for each case. From these values, the accuracy of solution for each case can be compared. That is, the smaller the value of V, the better the approximation is. As shown, the present approximation is much better than these by Levinson.[3]

The relationships between r and $\lambda^{1/4}$ for the first mode are listed in Table 2. As the value of r increases, the difference between the value of $\lambda^{1/4}$ for each case becomes large. The values of b for the dynamical case are similar to those for the statical case.

As stated previously, Murty[1] obtained the fundamental equation governing u_n on the assumption of $u = \Sigma_n u_n \eta^n$. But, variational conditions and boundary conditions are not always satisfied simultaneously, and, moreover, his equations do not satisfy the equilibrium conditions. Levinson[3,4] analyzed this problem assuming $Y = \eta^3$. In general, the condition $V_s = V_p$ is not satisfied for the case where Y is expressed by one term.

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